Mathematics 222B Lecture 24 Notes

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1 Existence of Minimizers for Lagrangian Actions

1.1 Hilbert's 19th problem

We will now set about giving an answer to Hilbert's 19th problem, which concerns minimizers for certain functionals in the calculus of variations:

$$\mathcal{F}[u] = \int_U L(Du, u, x)\, dx.$$

Under certain conditions (having to do with ellipticity of the Euler-Lagrange equation), there exists a minimizer. Hilbert's 19th problem asks about the regularity of such a minimizer. The minimizers that we find will a priori be in a class of rough functions, but in many situations, they will be solutions to some PDE and will have some smoothness.

This problem was solved by de Giorgi, then later by Nash, and later simplified by Moser. This is called de Giorgi-Nash-Moser theory. Today we will discuss existence, and next time, we will discuss regularity. Since we lost a lecture, we will not have time to discuss our last topic, which is hyperbolic PDEs which arise from calculus of variations. A good reference for this missing topic is *Lectures on nonlinear wave equations* by J. Luk.

1.2 Coercivity

We will basically follow the exposition in Section 8.2 of Evans. Consider a Lagrangian action functional

$$\mathcal{F}[u] = \int_U L(Du, u, x)\, dx.$$

We define the admissible class of functions u we want to minimizer over will be $\mathscr{A} = \{u \in W^{1,q}(U) : u|_{\partial U} = g\}$. The problem is to find

$$\operatorname*{arg\,min}_{u \in \mathscr{A}} \mathcal{F}[u].$$

We will look for "natural" conditions on L that would guarantee the existence of a minimizer. One pathology that may arise is that $\mathcal{F}$ could decay to 0 if we go to infinity in some direction, so we assume the following condition.

Definition 1.1. The action $\mathcal{F}$ is **coercive** if

$$L(p, z, x) \geq c|p|^q - \beta$$

for some constants $c, \beta > 0$ and $1 < q < \infty$.

Coercivity implies that

$$\begin{aligned}\mathcal{F}[u] &= \int_U L(Du, u, x)\,dx \\ &\geq c\int |Du|^q\,dx + \beta|U|.\end{aligned}$$

Using a Poincaré inequality, we can show that $\int_U |Du|^q\,dx$ controls the $W^{1.q}$ norm. In general, we should first determine the correct q from the action, which then specifies $\mathscr{A}$ accordingly.

1.3 Obstacles to convergence of a minimizing sequence

Let $\ell = \inf_{u\in\mathscr{A}} \mathcal{F}[u]$. There exists a sequence u_k such that $\mathcal{F}[u_k] \searrow \ell$. We want to say that

1. $u_k \to u \in \mathscr{A}$ for some u.
2. $\mathcal{F}[u_k] \to \mathcal{F}[u] = \ell$.

Then u will be a minimizer. In a finite dimensional setting, if we have compactness, we should actually assume that condition 1 is satisfied by a subsequence. But in fact, for $u_k \in \mathscr{A}$, both these conditions fail.

1. Failure of 1: From coercivity and a Poincaré inequality,

$$\begin{aligned}\|u_k\|_{W^{1,q}} &\lesssim \|Du_k\|_{L^q(U)} \\ &\lesssim \mathcal{F}[u_k] + \beta \\ &< \ell + \beta + 1.\end{aligned}$$

But there does not in general exist a convergent subsequence in $W^{1,q}$. Here are two ideas that may help us to proceed.

- Rellich-Kondrachov compactness tells us that there exists a subsequence $u_k \to u$ in $L^q(U)$.
- (weak compactness) Since $1 < q < \infty$, there exists a subsequence with $u_k \to u$ weakly in $W^{1,q}(U)$ (that is, $Du_k \to Du$ weakly in $L^q(U)$).

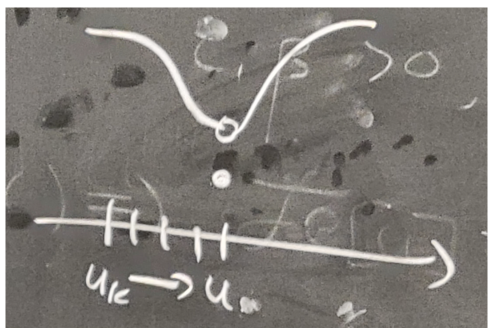
Without loss of generality, we may assume these are the same subsequence.

2. Failure of 2: Because $Du_k \to Du$ weakly, to ensure that $\mathcal{F}[u_k] \to \mathcal{F}[u]$, we need some sort of continuity of $\mathcal{F}$ under (sequential) weak convergence. It turns out that this is way too restrictive; weak convergence plays very well with linear operators but is in general badly behaved for nonlinear operators. As an example, $e^{ikx} \to 0$ weakly in $\mathcal{D}'(\mathbb{R}^d)$ as $k \to \infty$. On the other hand, $z\overline{z}|_{z=e^{ikx}} = 1 \not\to 0$, so even the simplest nonlinearity can cause issues.

   The fix here is to realize that we only need "half" of the continuity property because $\mathcal{F}[u_k] \searrow \mathcal{F}[u]$.

   Definition 1.2. A function f is **(sequentially) weak lower semicontinuous (LSC)** if for $u_k \to u$ weakly in $W^{1,q}(U)$ (i.e. $Du_k \to u$ weakly in $L^q(U)$ and $u_k \to u$ in $L^q(U)$), then

$$\liminf_{k\to\infty} \mathcal{F}[u_k] \geq \mathcal{F}[u].$$

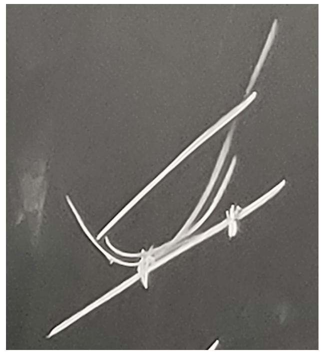


   Now, the question is: what is a natural condition on L that guarantees weak LSC of $\mathcal{F}$ on $W^{1,q}(U)$. The answer turns out to be convexity of L in p (Evans motivates this by looking at the Hessian of L):

$$\frac{\partial^2}{\partial p_j \partial p_k} L(p, z, x) \succeq 0 \qquad \forall p, z, x$$

   or equivalently,

$$L(p, z, x) \geq L(p_0, z, x) + D_p L(p_0, z, x) \cdot (p - p_0).$$



This is also equivalent to

$$L(\theta p_1 + (1-\theta)p_2, z, x) \leq \theta L(p_1, z, x) + (1-\theta)L(p_2, z, x).$$

Example 1.1. $L = |p|^q$ is convex for $q > 1$.

We will show that this convexity implies weak LSC for $\mathcal{F}$.[1]

1.4 Lower semicontinuity of the action

Here is the key theorem.

Theorem 1.1. *Assume L is convex in p, and assume coercivity: $L(p, z, x) \geq c|p|^q + \beta$. Then*

$$\mathcal{F}[u] = \int_U L(Du, u, x)\, dx$$

on $\mathcal{W}^{1,q}$ is weak LSC.

Proof. Assume without loss of generality that $\beta = 0$ (by replacing L by $L + \beta$). Take $\{u_k\} \in W^{1,q}(U)$ such that $Du_k \to Du$ weakly in L^q and $u_k \to u$ in $L^q(U)$. This is, up to subsequences, equivalent to $u_k \to u$ weakly in $W^{1,q}(U)$. Also passing to a subsequence, we can assume that $\mathcal{F}[u_k] \to \ell$. The goal is to show that $\ell \geq \mathcal{F}[u]$.

To handle nonlinear expressions in u_k, we use Egorov's theorem. Fix $\varepsilon > 0$. By Egorov's theorem, there exists a set G_ε such that

1. $|U \setminus G_\varepsilon| < \varepsilon$,
2. $u_k \to u$ uniformly on G_ε (up to a subsequence).

[1] It can be shown that these are actually eqiuvalent conditions.

Also, define $H_\varepsilon = \{x \in U : |u| < 1/\varepsilon, |Du| \leq 1/\varepsilon\}$. By the monotone convergence theorem, we can arrange that $|U \setminus H_\varepsilon| \lesssim \varepsilon$. On $A_\varepsilon := G_\varepsilon \cap H_\varepsilon$, we have property 2 and $|U \setminus A_\varepsilon| \lesssim \varepsilon$. Now

$$\mathcal{F}[u_k] = \int_U L(Du_k u_k, x)\, dx$$

Since $L \geq c|p|^q$, it is ≥ 0. So we can shrink the domain of integration.

$$\begin{aligned} &\geq \int_{A_\varepsilon} L(Du_k, u_k, x)\, dx \\ &\geq \int_{A_\varepsilon} \underbrace{L(Du, u_k, x)}_{I} + \underbrace{D_p L(Du, u_k, x)(Du_k - Du)}_{II}\, dx. \end{aligned}$$

Take $k \to \infty$, so the left hand side converges to ℓ. By uniform convergence (and continuity of L in p, which we assume),

$$\int_{A_\varepsilon} I\, dx \to \int_{A_\varepsilon} L(du, u, x)\, dx.$$

For the other term,

$$\begin{aligned} \int_{A_\varepsilon} II\, dx = \int_{A_\varepsilon} &\underbrace{(D_p L(Du, u_k, x) - D_p L(Du, u, x))}_{\to 0 \text{ unif.}} \cdot \underbrace{(Du_k - Du)}_{\|\cdot\|_{L^q} \lesssim 1}\, dx \\ &+ \int_{A_\varepsilon} D_p(Du, u, x,) \cdot (Du_k - Du)\, dx, \end{aligned}$$

and the latter term goes to 0 thanks to the weak convergence of $Du_k \to Du$. Thus, we have

$$\ell \geq \int_{A_\varepsilon} L(Du, u, x)\, dx.$$

Let $\varepsilon \to 0$ so that "$|U \setminus A_\varepsilon| \to 0$." This gives

$$\ell \geq \int_U L(Du, u, x)\, dx,$$

as desired. □

Remark 1.1. We have been omitting some regularity assumptions on L.

1.5 Proof of existence of minimizers

Theorem 1.2. *In addition to regularity assumptions on L, assume that L is convex in p and $L \geq c|p|^q + \beta$. Consider $\mathscr{A} = \{u \in W^{1,q}(U) : u|_{\partial U} = g\}$ and the action*

$$\mathcal{F}[u] = \int_U L(Du, u, x)\, dx.$$

There exists a minimizer u for $\mathcal{F}[u]$ in $\mathscr{A}$.

Remark 1.2. Uniqueness and regularity conditions require more assumptions on L, which upgrade this convexity property.

Proof. Take a minimizing sequence u_k such that $\mathcal{F}[u_k] \searrow \ell$, where $\ell = \inf_{u \in \mathscr{A}} \mathcal{F}[u] < \infty$ (if this is $\ell = +\infty$, there is nothing to prove). By this and coercivity, $\|Du_k\|_{L^q(U)} \lesssim 1$. There exists some extension $\widetilde{g} \in W^{1,q}$ such that $\widetilde{g}|_{\partial U} = g$, so we can consider $u_k - \widetilde{g} \in W_0^{1,q}(U)$. A Poincaré inequality gives

$$\begin{aligned} \|u_k - \widetilde{g}\|_{W^{1,q}(U)} &\lesssim \|Du_k - D\widetilde{g}\|_{L^q(U)} \\ &\lesssim 1. \end{aligned}$$

By weak compactness of the norm-unit ball in $L^q(U)$, up to a subsequence, we may assume $Du_k \to Du$ weakly in $L^q(U)$. By Rellich-Kondrachov compactness, up to a subsequence, $u_k \to u$ in $L^q(U)$. Now apply the weak LSC theorem to get that

$$\ell = \inf_{v \in \mathscr{A}} \mathcal{F}[v] \leq \mathcal{F}[u] \leq \ell.$$

This gives $\mathcal{F}[u] = \ell$. □

Theorem 1.3. *Let L satisfy*

$$|L| \leq c(|p|^q + |z|^q + 1), \quad |D_p L| \leq c(|p|^{q-1} + |z|^{q-1} + 1), \quad |D_z L| \leq C(|p|^{q-1} + |z|^{q-1} + 1).$$

Then any minimizer u for $\mathcal{F}[u]$ in $\mathscr{A}$ is a weak solution to the Euler-Lagrange equation. That is,

$$\int_U (\partial_{p_j} L(Du, u, x) \partial_{x^j} v + \partial_z L(Du, u, x) v) \, dx \qquad \forall v \in W_0^{1,p}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

See Evans for the proof.